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Non-commutative Chern-Simons terms and the non-commutative vacuum

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Abstract: It is pointed out that the space noncommutativity parameters $\theta^{\mu\nu}$ in non-commutative gauge theory can be considered as a set of superselection parameters, in analogy with the θ-angle in ordinary gauge theories. As such, they do not need to enter explicitly into the action. A simple generic formula is then suggested to reproduce the Chern-Simons action in non-commutative gauge theory, which reduces to the standard action in the commutative limit but in general implies a cascade of lower-dimensional Chern-Simons terms. The presence of these terms in general alters the vacuum structure of the theory and non-standard gauge theories can emerge around the new vacua.

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Contents

1. Introduction

Chern-Simons actions have appeared in various contexts as topological terms in the action for gauge fields in odd (spacetime) dimensions [1]. Their densities are essentially defined as the gauge-field dependent differential form whose exterior derivative equals $\mathrm{tr}\,\mathrm{F}^{n+1}$, with F the field strength. Specifically, we define the gauge field one-form and the field strength two-form as

$$\mathrm{A} = iA_\mu dx^\mu\,, \qquad \mathrm{F} = \mathrm{dA} + \mathrm{A}^2 = \frac{i}{2}\Big(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\Big)dx^\mu dx^\nu\,. \tag{1.1}$$

The Chern-Simons action S_{2n+1} is the integral of the $2n+1$-form C_{2n+1} satisfying

$$\mathrm{dC}_{2n+1} = \mathrm{tr}\,\mathrm{F}^{n+1}\,. \tag{1.2}$$

By virtue of (1.2) and the gauge invariance of $\mathrm{tr}\,\mathrm{F}^n$ it follows that S_{2n+1} is gauge invariant up to total derivatives, since, if δ stands for an infinitesimal gauge transformation,

$$\mathrm{d}\delta\mathrm{C}_{2n+1} = \delta\mathrm{dC}_{2n+1} = \delta\,\mathrm{tr}\,\mathrm{F}^n = 0\,, \qquad \text{so } \delta C_{2n+1} = \mathrm{d}\Omega_{2n}\,. \tag{1.3}$$

The integrated action is therefore invariant under infinitesimal gauge transformations. Large gauge transformations may lead to an additive change in the action and they usually imply a quantization of its coefficient [1] (for a review see [2]). As a result, the equations of motion derived from this action are gauge covariant and read

$$\frac{\delta S_{2n+1}}{\delta \mathrm{A}} = \frac{\delta}{\delta \mathrm{A}}\int \mathrm{C}_{2n+1} = (n+1)\mathrm{F}^n\,. \tag{1.4}$$

We shall consider (1.4) as the defining relation for C_{2n+1}.

2. Non-commutative gauge theory

Similar actions can be defined in non-commutative theories [3]–[7]. These are gauge theories based upon non-commutative spaces [8, 9]. The coordinates X^μ of such spaces obey the commutation relations

$$[X^\mu, X^\nu] = i\theta^{\mu\nu}\,, \qquad \mu,\nu = 1,\dots d\,. \tag{2.1}$$

The antisymmetric two-tensor $\theta^{\mu\nu}$ is usually taken to commute with all X^μ and is, thus, a set of constant c-numbers. Its inverse

$$\omega_{\mu\nu} = (\theta^{-1})_{\mu\nu} \tag{2.2}$$

defines a constant two-form ω characterising the noncommutativity of the space. Clearly θ and ω can be brought into a canonical (Darboux) form by an appropriate orthogonal rotation of the space, breaking it into non-commutative two-dimensional subspaces. Some coordinates may, then, be commuting (odd-dimensional spaces necessarily have at least one commuting coordinate). In general, there will be $2n$ properly non-commuting coordinates X^α ($\alpha = 1,\dots 2n$) and $q = d - 2n$ commuting ones Y^i ($i = 1,\dots q$). In that case ω will be defined as the inverse of the projection $\bar\theta$ of θ on the fully non-commuting subspace:

$$\omega_{\alpha\beta} = (\bar\theta^{-1})_{\alpha\beta}\,, \qquad \omega_{i\mu} = 0\,. \tag{2.3}$$

The actual spacetime can be though of as a representation of the above operator algebra (2.1). For the commuting components Y^i this representation must necessarily be reducible (else the corresponding directions would effectively be absent, consisting of a single point); states are labeled by the values of these coordinates y^i, taken to be continuous. The rest of the space is represented by the tensor product of Heisenberg-Fock space representations (one for each two-dimensional non-commuting subspace $k = 1,\dots n$). In general, we can have the direct sum of N such irreducible components for each set of values y^i, labeled by an extra index $a = 1,\dots N$ (we shall take N not to depend on y^i). A complete basis for the states, then, can be

$$|n_1,\dots n_n; y^1,\dots y^q; a\rangle\,, \tag{2.4}$$

where n_k is the Fock (oscillator) excitation number of the k-th two-dimensional subspace.

Due to the reducibility of the above representation, the operators X^μ do not constitute a complete set. An important set of additional operators are translation (derivative) operators ∂_μ. These are defined through their action on X^μ, generating constant shifts:

$$[\partial_\mu, X^\nu] = \delta_\mu^\nu\,. \tag{2.5}$$

On the fully non-commutative subspace these are inner automorphisms generated by

$$\partial_\alpha = -i\omega_{\alpha\beta} X^\beta \,. \tag{2.6}$$

For the commutative coordinates, however, extra operators have to be appended, shifting the Casimirs Y^i and thus acting on the coordinates y^i as usual derivatives. There is yet another set of operators in the full representation space: G^r, $r = 1, \dots N^2$, the hermitian operators mixing the irreducible components a. The set of $X^\alpha, \partial_i, G^r$ is now complete.

The integral over space is defined as the trace in the full representation, normalized as:

$$\int d^d x = \int d^q y \, \mathrm{tr}' \sqrt{\det(2\pi\theta)} \; \mathrm{tr} \equiv \mathrm{Tr} \,, \tag{2.7}$$

where tr is the trace over the Fock spaces and tr′ is the trace over the degeneracy index $a = 1, \dots N$.

Gauge fields A_μ are arbitrary functions of the non-commutative coordinates X^μ and thus are (hermitian) operators acting on the full representation space. Since they do not depend on ∂_i they cannot shift the values of y^i, while they act nontrivially on the fully non-commuting subspace. They have effectively become big matrices acting on the full Fock space with elements depending on the commuting coordinates. Derivatives of these fields are defined through the adjoint action of ∂_μ

$$\partial_\mu \cdot A_\nu = [\partial_\mu, A_\nu] \,. \tag{2.8}$$

Using the above formalism, gauge field theory can be built in a way analogous to the commuting case. Gauge transformations are unitary transformations in the full representation space. Restricting A_μ to depend on the coordinates only, as above, produces the so-called U(1) gauge theory. U(N) gauge theory can be obtained by relaxing this restriction and allowing A_μ to also be a function of the G^r and thus act on the index a (see also [10]). Further generalizations in which A_μ are also allowed to depend on ∂_i and act nontrivially on y^i, thus becoming completely general hermitian operators, are possible but are usually not considered.

We should mention that an alternative way to describe field theories on non-commutative spaces is by ordering the operators X_μ in the expressions for the fields in a specific way and thus establishing a one-to-one correspondence between functions of operators and ordinary functions. (The Weyl symmetric ordering is usually adopted.) The product of operators, then, is mapped to the so-called star-product of functions, while derivatives and integrals are the standard commutative ones. This approach has the advantage of bypassing some of the conceptual issues arising in non-commutative theories, but also the disadvatntage of obscuring their basic simplicity and some of their interesting structure. We shall employ the operator formalism in this paper.

3. Superselection of the non-commutative vacuum

The basic moral of the above discussion is that non-commutative gauge theory can be written in a universal way. In the operator formulation no special distinction needs be done between U(1) and U(N) theories, nor need gauge and spacetime degrees of freedom be treated distinctly. The fundamental operators of the theory are

$$D_\mu = -i\partial_\mu + A_\mu \tag{3.1}$$

corresponding to covariant derivatives. Under gauge transformations they transform covariantly:

$$D_\mu \longrightarrow U D_\mu U^{-1} \,. \tag{3.2}$$

Any lagrangian built entirely out of D_μ will lead to a gauge invariant action, since the trace will remain invariant under any unitary transformation. The standard Lorentz-Yang-Mills action is built by defining the field strength

$$F_{\mu\nu} = \partial_\mu \cdot A_\nu - \partial_\nu \cdot A_\mu + i[A_\mu, A_\nu] = i[D_\mu, D_\nu] - \omega_{\mu\nu} \tag{3.3}$$

and writing the standard action

$$S_{\mathrm{LYM}} = \frac{1}{4g^2}\,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4g^2}\,\mathrm{Tr}([D_\mu, D_\nu] + i\omega_{\mu\nu})^2 \tag{3.4}$$

(we used some metric tensor $g^{\mu\nu}$ to raise the indices of F). Note that the operators $\partial_\alpha\cdot$, understood to act in the adjoint on fields, commute, while the operators $\partial_\alpha = -i\omega_{\alpha\beta}X^\beta$ have a non-zero commutator equal to

$$[\partial_\alpha, \partial_\beta] = i\omega_{\alpha\beta}\,. \tag{3.5}$$

This explains the extra ω-term appearing in the definition of F through covariant derivative commutators.

As was pointed out in [11], however, one can just as well work with the action

$$\hat{S}_{\mathrm{LYM}} = \frac{1}{4g^2}\,\mathrm{Tr}\, \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} = -\frac{1}{4g^2}\,\mathrm{Tr}[D_\mu, D_\nu]^2\,. \tag{3.6}$$

Indeed, $\hat{S}$ differs from S by a term proportional to $\mathrm{Tr}\,\omega^2$, which is an irrelevant (infinite) constant, as well as a term proportional to $\omega^{\mu\nu}\,\mathrm{Tr}[D_\mu, D_\nu]$, which, being the trace of a commutator (a "total derivative"), does not contribute to the equations of motion. The two actions then lead to the same classical theory. The equations of

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motion for the operators D_μ are

$$[D_\mu, [D_\mu, D_\nu]] = 0\,. \tag{3.7}$$

Apart from the trivial solution $D_\mu = 0$ this has as solution all operators with c-number commutators, satisfying

$$[D_\mu, D_\nu] = -i\omega_{\mu\nu} \tag{3.8}$$

for some ω. This is the classical "non-commutative vacuum", where $D_\mu = -i\partial_\mu$, and expanding D_μ around this vacuum leads to non-commutative gauge theory. Quantum mechanically $\omega_{\mu\nu}$ are superselection parameters and the above vacuum is stable. To see this, assume that the time direction is commutative and consider the collective mode

$$D_\alpha = -i\lambda_{\alpha\beta}\partial_\beta \tag{3.9}$$

with $\lambda_{\alpha\beta}$ parameters depending only on time. This mode would change the non-commutative vacuum while leaving the gauge field part of D_α unexcited. ω gets modified into

$$\omega'_{\mu\nu} = \lambda_{\mu\alpha}\omega_{\alpha\beta}\lambda_{\beta\nu}\,. \tag{3.10}$$

The action implies a quartic potential for this mode, with a strength proportional to $\mathrm{Tr}\,1$, and a kinetic term proportional to $\mathrm{Tr}\,\partial_\alpha\partial_\beta$. (There is also a gauge constraint which does not alter the qualitative dynamical behavior of λ.) Both potential and kinetic terms are infinite, and to regularize them we should truncate each Fock space trace up to some highest state Λ, corresponding to a finite volume regularization (the area of each non-commutative two-dimensional subspace has effectively become Λ). It can be seen that the potential term would grow as Λ^n while the kinetic term would grow as Λ^{n+1}. Thus the kinetic term dominates; the above collective degrees of freedom acquire an infinite mass and will remain "frozen" to whatever initial value they are placed, in spite of the non-trivial potential. (This is analogous to the θ-angle of the vacuum of four-dimensional non-abelian gauge theories: the vacuum energy depends on θ which is still superselected.) Quantum mechanically there is no interference between different values of λ and we can fix them to some c-number value, thus fixing the noncommutativity of space. This phenomenon is similar to symmetry breaking, but with the important difference that the potential is not flat along changes of the "broken" vacuum, and consequently there are no Goldstone bosons.

In conclusion, we can start with the action (3.6) as the definition of our theory, where D_μ are arbitrary operators (matrices) in some space. Gauge theory is then defined as a perturbation around a (stable) classical vacuum. Particular choices of this vacuum will lead to standard non-commutative gauge theory, with $\theta^{\mu\nu}$ and N appearing as vacuum parameters.

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4. Non-commutative Chern-Simons action

We are now set to define Chern-Simons actions. To this end, we shall define the usual basis of one-forms dx^μ as a set of formal anticommuting parameters with the property

$$dx^\mu dx^\nu = -dx^\nu dx^\mu \,, \qquad dx^{\mu_1} \cdots dx^{\mu_d} = \epsilon^{\mu_1 \cdots \mu_d} \,. \tag{4.1}$$

Topological actions do not involve the metric tensor and can be written as integrals of d-forms. The only dynamical objects available in our theory are D_μ and thus the only form that we can write is

$$\mathrm{D} = i dx^\mu D_\mu = \mathrm{d} + \mathrm{A} \,, \tag{4.2}$$

where we defined the exterior derivative and gauge field one-forms

$$\mathrm{d} = dx^\mu \partial_\mu \,, \qquad \mathrm{A} = i dx^\mu A_\mu \tag{4.3}$$

(note that both D and A are antihermitian). The action of the exterior derivative d on an operator p-form H, $\mathrm{d} \cdot \mathrm{H}$, yields the $p+1$-form $dx^\mu [\partial_\mu, \mathrm{H}]$ and is given by

$$\mathrm{d} \cdot \mathrm{H} = \mathrm{dH} - (-)^p \mathrm{Hd} \,. \tag{4.4}$$

In particular, on the gauge field one-form A it acts as

$$\mathrm{d} \cdot \mathrm{A} = \mathrm{dA} + \mathrm{Ad} \,. \tag{4.5}$$

Correspondingly, the covariant exterior derivative of H is

$$\mathrm{D} \cdot \mathrm{H} = \mathrm{DH} - (-)^p \mathrm{HD} \,. \tag{4.6}$$

As a result of the noncommutativity of the operators ∂_μ, the exterior derivative operator is not nilpotent but rather satisfies

$$\mathrm{d}^2 = \omega \,, \qquad \omega = \frac{i}{2} dx^\mu dx^\nu \omega_{\mu\nu} \,. \tag{4.7}$$

We stress, however, that $\mathrm{d}\cdot$ is still nilpotent since ω commutes with all operator forms:

$$\mathrm{d} \cdot \mathrm{d} \cdot \mathrm{H} = [\mathrm{d}, [\mathrm{d}, \mathrm{H}]_\mp]_\pm = \pm[\omega, \mathrm{H}] = 0 \,. \tag{4.8}$$

The two-form $\hat{\mathrm{F}} = \frac{i}{2} dx^\mu dx^\nu \hat{F}_{\mu\nu}$ is simply

$$\hat{\mathrm{F}} = \mathrm{D}^2 = \frac{1}{2} \mathrm{D} \cdot \mathrm{D} = \omega + \mathrm{dA} + \mathrm{Ad} + \mathrm{A}^2 = \omega + \mathrm{F} \,, \tag{4.9}$$

where $\mathrm{F} = \frac{i}{2} dx^\mu dx^\nu F_{\mu\nu}$ is the conventionally defined field strength two-form.

The most general d-form that we can write involves arbitrary combinations of D and ω. If, however, we adopt the view that ω should arise as a superselection (vacuum) parameter and not as a term in the action, the unique form that we can write is D^d and the unique action

$$\hat{S}_d = \frac{d+1}{2d} \,\mathrm{Tr}\; \mathrm{D}^d = \mathrm{Tr}\; \mathrm{C}_d\,. \tag{4.10}$$

This is the Chern-Simons action. The coefficient was chosen to conform with the commutative definition (1.4), as will be seen shortly. In even dimensions $\hat{S}_d$ reduces to the trace of a commutator $\mathrm{Tr}[\mathrm{D}, \mathrm{D}^{d-1}]$, a total derivative that does not affect the equations of motion and corresponds to a topological term. In odd dimensions it becomes a non-trivial action.

$\hat{S}_d$ is by construction gauge invariant. To see that it also satisfies the defining property of a Chern-Simons form (1.4) is almost immediate: $\delta/\delta\mathrm{A} = \delta/\delta\mathrm{D}$ and thus, for $d = 2n+1$:

$$\frac{\delta}{\delta \mathrm{A}}\,\mathrm{Tr}\;\mathrm{D}^{2n+1} = (2n+1)\,\mathrm{D}^{2n} = (2n+1)\,\hat{\mathrm{F}}^n\,. \tag{4.11}$$

So, with the chosen normalization in (4.10) we have the defining condition (1.4) with $\hat{\mathrm{F}}$ in the place of F. What is less obvious is that $\hat{S}_D$ can be written entirely in terms of F and A and that, for commutative spaces, it reduces to the standard Chern-Simons action. To achieve that, one must expand C_d in terms of d and A, make use of the cyclicity of trace and the condition $\mathrm{d}^2 = \omega$ and reduce the expressions into ones containing $\mathrm{dA} + \mathrm{Ad}$ rather than isolated d's. The condition

$$\mathrm{Tr}\,\omega^n \mathrm{d} = 0 \tag{4.12}$$

which is a result of the fact that ∂_μ is off-diagonal for both commuting and non-commuting dimensions, can also be used to get rid of overall constants. This is a rather involved procedure for which we have no algorithmic approach. In the appendix we will illustrate the case $d = 7$, which presents the full range of complexity in terms of reduction to ordinary terms. Note, further, that the use of the cyclicity of trace implies that we dismiss total derivative terms (traces of commutators). Such terms do not affect the equations of motion. For $d = 1$ the result is simply

$$\hat{S}_1 = \mathrm{Tr}\,\mathrm{A} \tag{4.13}$$

which is the "abelian" one-dimensional Chern-Simons term. For $d = 3$ we obtain

$$\hat{S}_3 = \mathrm{Tr}\left(\mathrm{AF} - \frac{1}{3}\mathrm{A}^3\right) + 2\,\mathrm{Tr}(\omega\mathrm{A})\,, \tag{4.14}$$

where we used the fact that $\mathrm{Tr}[\mathrm{A}(\mathrm{dA}+\mathrm{Ad})] = 2\,\mathrm{Tr}(\mathrm{A}^2\mathrm{d})$. The first term is the non-commutative version of the standard three-dimensional Chern-Simons term, while the second is a lower-dimensional Chern-Simons term involving explicitly ω.

We can get the general expression for $\hat{S}_d$ by referring to the defining relation. This reads

$$\frac{\delta}{\delta \mathrm{A}}\hat{S}_{2n+1} = (n+1)\hat{\mathrm{F}}^n = (n+1)(\mathrm{F}+\omega)^n = (n+1)\sum_{k=0}^{n}\binom{n}{k}\,\omega^{n-k}\mathrm{F}^k \tag{4.15}$$

and by expressing F^k as the A-derivative of the standard Chern-Simons action S_{2k+1} we get

$$\frac{\delta}{\delta \mathrm{A}}\left\{\hat{S}_{2n+1}-\sum_{k=0}^{n}\binom{n+1}{k+1}\omega^{n-k}S_{2k+1}\right\}=0\,. \tag{4.16}$$

So the expression in brackets must be a constant, easily seen to be zero by setting $\mathrm{A}=0$. We therefore have

$$\hat{S}_{2n+1}=\sum_{k=0}^{n}\binom{n+1}{k+1}\mathrm{Tr}\,\omega^{n-k}\mathrm{C}_{2k+1}\,. \tag{4.17}$$

We observe that we get the $(2n+1)$-dimensional Chern-Simons action plus all lower-dimensional actions with tensors ω inserted to complete the dimensions. Each term is separately gauge invariant and we could have chosen to omit them, or include them with different coefficient. It is the specific combination above, however, that has the property that it can be reformulated in a way that does not involve ω explicitly. The standard Chern-Simons action can also be written in terms of D alone by inverting (4.17):

$$S_{2n+1}=(n+1)\,\mathrm{Tr}\int_0^1(t^2\mathrm{D}^2-\omega)^n dt=\mathrm{Tr}\sum_{k=0}^{n}\binom{n+1}{k+1}\frac{k+1}{2k+1}(-\omega)^{n-k}\mathrm{D}^{2k+1}\,. \tag{4.18}$$

We also point out a peculiar propery of the Chern-Simons form $\hat{\mathrm{C}}_{2n+1}$. Its covariant derivative yields $\hat{\mathrm{F}}^{n+1}$:

$$\mathrm{D}\cdot\hat{\mathrm{C}}_{2n+1}=\mathrm{D}\hat{\mathrm{C}}_{2n+1}+\hat{\mathrm{C}}_{2n+1}\mathrm{D}=\frac{2n+2}{2n+1}\,\hat{\mathrm{F}}^{n+1}\,. \tag{4.19}$$

A similar relation holds between C_d (understood as the form appearing inside the trace in the right-hand side of (4.18)) and F. Clearly the standard Chern-Simons form does not share this property. Our C_d differs from the standard one by commutators that cannot all be written as ordinary derivatives (such as, e.g. $[\mathrm{d},\mathrm{dA}]$). These unconventional terms turn C_d into a covariant quantity that satisfies (4.19).

We remark here that a construction reminiscent of the above exists in string field theory [12], where the action is written as the integral of $(Q+A)^3$, with Q the BRST charge. We also remark that actions similar to the subleading terms in (4.17) were examined a while ago in the commutative case [13]. In particular, the five-dimensional action ωS_3, with ω a Kähler form, was considered as a way to construct conformally invariant theories in four dimensions.

5. New vacua

We conclude with the observation that the addition of the Chern-Simons action changes the equations of motion of the theory and thus it modifies the non-commutative vacuum. As an example, in three dimensions the equations of motion read

$$[D_\nu,[D_\nu,D_\mu]]=i\lambda\epsilon_{\mu\nu\rho}[D_\nu,D_\rho] \tag{5.1}$$

with λ a coupling constant. (A similar equation was obtained from a different point of view in [14].) For euclidean metric this has finite dimensional solutions. Specifically,

$$D_\mu = -\lambda J_\mu \,, \qquad J^2 = j(j+1)\,, \tag{5.2}$$

where J_μ are three SU(2) matrices in an arbitrary spin representation. Correspondingly, for Minkowski metric we get as solutions all the (infinite-dimensional) unitary representations of SU(1, 1). The resulting gauge theories deserve further investigation.

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A. Calculation of S_7

We shall calculate $\hat{S}_7$ starting from the definition

$$\hat{S}_7 = \frac{4}{7}\,\mathrm{Tr}\,\mathrm{D}^7 = \frac{4}{7}\,\mathrm{Tr}(\mathrm{d}+\mathrm{A})^7\,. \tag{A.1}$$

Expanding $(\mathrm{d}+\mathrm{A})^7$ inside the trace we will get all combinations of d and A positioned in a cyclic arrangement. Clearly if two d are adjacent on the cycle they will contract to ω and will contribute towards lower-order Chern-Simons terms, of order ω or more. Concentrating on the leading term of order ω^0, the relevant combinations are

$$\hat{S}_7 = \frac{4}{7}\,\mathrm{Tr}\left(\mathrm{A}^7 + 7\mathrm{A}^6\mathrm{d} + 7\mathrm{A}^4\mathrm{dAd} + 7\mathrm{A}^3\mathrm{dA}^2\mathrm{d} + 7\mathrm{A}^2\mathrm{dAdAd}\right) + \mathcal{O}(\omega)\,. \tag{A.2}$$

From now on all relations will be understood as holding to order ω^0, omitting terms of order ω without explicitly writing $+\mathcal{O}(\omega)$.

The second term can be expressed as

$$\mathrm{Tr}(\mathrm{A}^6\mathrm{d}) = \frac{1}{2}\,\mathrm{Tr}[\mathrm{A}^5(\mathrm{dA}+\mathrm{Ad})] = \frac{1}{2}\,\mathrm{Tr}\left[\mathrm{A}^5(\mathrm{F}-\mathrm{A}^2)\right]. \tag{A.3}$$

To reduce the next two terms we observe

$$(\mathrm{F}-\mathrm{A}^2)^2 = (\mathrm{dA}+\mathrm{Ad})^2 = \mathrm{dAdA} + \mathrm{dA}^2\mathrm{d} + \mathrm{AdAd} \tag{A.4}$$

and

$$\begin{aligned}(\mathrm{F}-\mathrm{A}^2)\mathrm{A}(\mathrm{F}-\mathrm{A}^2) &= (\mathrm{dA}+\mathrm{Ad})\mathrm{A}(\mathrm{dA}+\mathrm{Ad}) \\ &= \mathrm{dA}^2\mathrm{dA} + \mathrm{dA}^3\mathrm{d} + \mathrm{AdAdA} + \mathrm{AdA}^2\mathrm{d}\,. \end{aligned} \tag{A.5}$$

Tracing the previous two relations with A^3 and A^2, respectively we get

$$\begin{aligned}\mathrm{Tr}\left[\mathrm{A}^3(\mathrm{F}-\mathrm{A}^2)^2\right] &= 2\,\mathrm{Tr}(\mathrm{A}^4\mathrm{dAd}) + \mathrm{Tr}(\mathrm{A}^3\mathrm{dA}^2\mathrm{d})\,,\\ \mathrm{Tr}\left[\mathrm{A}^2(\mathrm{F}-\mathrm{A}^2)\mathrm{A}(\mathrm{F}-\mathrm{A}^2)\right] &= 3\,\mathrm{Tr}(\mathrm{A}^3\mathrm{dA}^2\mathrm{d}) + \mathrm{Tr}(\mathrm{A}^4\mathrm{dAd})\end{aligned}\tag{A.6}$$

and solving for the two terms in the right-hand side we obtain

$$\begin{aligned}\mathrm{Tr}(\mathrm{A}^4\mathrm{dAd}) &= \frac{3}{5}\,\mathrm{Tr}\left[\mathrm{A}^3(\mathrm{F}-\mathrm{A}^2)^2\right] - \frac{1}{5}\,\mathrm{Tr}\left[\mathrm{A}^2(\mathrm{F}-\mathrm{A}^2)\mathrm{A}(\mathrm{F}-\mathrm{A}^2)\right],\\ \mathrm{Tr}(\mathrm{A}^3\mathrm{dA}^2\mathrm{d}) &= \frac{2}{5}\,\mathrm{Tr}\left[\mathrm{A}^2(\mathrm{F}-\mathrm{A}^2)\mathrm{A}(\mathrm{F}-\mathrm{A}^2)\right] - \frac{1}{5}\,\mathrm{Tr}\left[\mathrm{A}^3(\mathrm{F}-\mathrm{A}^2)^2\right].\end{aligned}\tag{A.7}$$

Finally, to obtain the last rem we observe

$$(\mathrm{F}-\mathrm{A}^2)^3 = (\mathrm{dA}+\mathrm{Ad})^3 = \mathrm{dAdAdA} + \mathrm{dAdA}^2\mathrm{d} + \mathrm{dA}^2\mathrm{dAdA} + \mathrm{AdAdAd}\tag{A.8}$$

and tracing with A

$$\mathrm{Tr}(\mathrm{A}^2\mathrm{dAdAd}) = \frac{1}{4}\,\mathrm{Tr}\left[\mathrm{A}(\mathrm{F}-\mathrm{A}^2)^3\right].\tag{A.9}$$

Putting together all the terms we obtain $\hat{S}_7 = S_7 + \mathcal{O}(\omega)$ as

$$S_7 = \mathrm{Tr}\left(\mathrm{AF}^3 - \frac{2}{5}\mathrm{A}^3\mathrm{F}^2 - \frac{1}{5}\mathrm{A}^2\mathrm{FAF} + \frac{1}{5}\mathrm{A}^5\mathrm{F} - \frac{1}{35}\mathrm{A}^7\right)\tag{A.10}$$

which is the standard 7-dimensional Chern-Simons action.

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